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# Proprietates triangulorum, quorum anguli certam inter se tenent rationem

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Euler, Leonhard, "Proprietates triangulorum, quorum anguli certam inter se tenent rationem" (1767). *Euler Archive - All Works*. 324.  
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# PROPRIETATES TRIANGVLORVM, QVORVM ANGVLI CERTAM INTER SE TENENT RA- TIONEM.

Auctore

L. EVLERO.

**I**nter veritates geometricas eae potissimum atten-  
tione sunt dignae, quarum demonstratio ita est  
recondita, vt analyticae inuestigationi vix vllus lo-  
cus relinqui videatur. Quae enim ita sunt compara-  
tae, vt formula analytica facile comprehendi queant,  
omnino superfluum foret, memoriam earum recor-  
datione fatigare: ad quod genus plurimae sectionum  
conicarum proprietates sunt referendae, quarum  
plerumque ingens multitudo vnica formula analyti-  
ca includi potest. Elementares autem figurarum  
proprietates eo maiori cura memoriae sunt mandan-  
dae, quod analysis ad eas non perducatur, sed iis  
potius ad altiora tendens superstrui debeat. Nescio,  
an proprietates triangulorum, - quas hic euoluere  
constitui, elementaribus sint annumerandae, nec ne?  
Si enim ad earum demonstrationes geometricas spe-  
ctemus, eae ita sunt intricatae, vt in elementis  
locum vix inuenire queant: tum vero etiam, quod  
hic imprimis est obseruandum, ne analysis quidem  
satis videtur idonea, ad earum veritatem stabilien-  
dam;

dam ; quamobrem hanc speculationem attentioni geometrarum commendare non dubito.

Occasionem autem, haec perscrutandi, mihi prae-  
buit prima quasi triangulorum proprietas elementa-  
ris, qua novimus, si duo anguli fuerint inter se  
aequales, etiam duo latera, ipsis scilicet opposita, in-  
ter se aequalia esse futura. Quemadmodum ergo  
hoc casu ex data angulorum conditione certa relatio  
laterum sequitur, ita generatim affirmare licet,  
quoties in triangulo certa quaedam ratio inter duos  
angulos datur, inde necessario quoque certam quan-  
dam relationem inter latera determinari. Ex quo  
haec nascitur quaestio: *Si in triangulo, cuius anguli  
sint  $\alpha$ ,  $\beta$ ,  $\gamma$ , latera iis opposita litteris  $a$ ,  $b$ ,  $c$   
designentur, haecque conditio detur, ut sit  $\alpha:\beta = m:n$ ,  
relationem inter latera  $a$ ,  $b$ ,  $c$  inde ortam inuestigare?*  
Problema hoc statim ac ratio data  $m:n$  tantillum  
assumitur complicata, analytice tractatum in taedio-  
sissimos calculos praecipitare tentanti mox patebit:  
sin autem a casu simplicissimo, quo  $\beta = \alpha$ , et  $b = a$ ,  
incipientes, continuo ad magis compositos ordine pro-  
grediamur, egregiam tandem progressionis legem  
obseruare licebit, quae eo magis est notatu digna,  
quod per solam inductionem sit inuenta, vixque  
demonstrationem admittere videatur.

### Problema I.

Tab. I. I. Si in triangulo ABC fuerit ang. B = 2 ang. A,  
Fig. 2- inter eius latera  $AB = c$ ,  $AC = b$  et  $BC = a$  rela-  
tionem inde oriundam inuestigare.

Solutio.

### Solutio.

Angulo B per rectam BD bisecto, erit triangulum ADB isosceles, et triangulum BCD toti ACB simile,

vnde fit  $AC:BC=AB:BD=BC:CD$ ,

$$\text{feu } b:a=c:\frac{ac}{b}=a:\frac{aa}{b}.$$

Ergo  $BD=\frac{ac}{b}$  et  $CD=\frac{aa}{b}$ ; hinc  $AD=b-\frac{aa}{b}$ .

At ob  $BD=AD$  habebimus  $ac=bb-aa$ , qua ergo aequatione continetur relatio quaesita inter latera trianguli, quae est vel  $(AC+BC)(AC-BC)=AB.BC$  vel  $AC^2=BC(AB+BC)$ .

### Coroll. 1.

2. Ultima aequatio facilem hanc suppeditat demonstrationem formulae inuentae; producto enim latere AB in C, vt sit  $BE=BC$ , erit angulus E semissis ipsius ABC, ideoque ipsi A aequalis, vnde triangula isoscelia ACE et CBE erunt similia; hinc  $AE:AC=CE:BC$ , seu  $AB+BC:AC=AC:BC$ .

### Coroll. 2.

3. Vicissim ergo, quoties inter latera trianguli ABC haec relatioprehenditur, vt fit  $AC^2=BC.(AB+BC)$  seu  $bb=aa+ac$ , toties concludi oportet, angulum ABC esse duplum anguli BAC.

## Scholion.

4. Haec inuersa propositio, etsi eius veritas ex praecedente necessario sequitur, tamen non ita facile geometricè demonstratur. Si scilicet fuerit  $AC^2 = AB \cdot BC + BC^2$ , ostendendum est, fore angulum A semissem anguli ABC. Hunc in finem demisso ex C in AB perpendicularo CP, ex elementis constat, esse  $AC^2 = AB^2 + BC^2 - 2AB \cdot BP$ ; cum igitur sit  $AC^2 = AB \cdot BC + BC^2$ , erit  $BC^2$  vtrinque auferendo  $AB^2 - 2AB \cdot BP = AB \cdot BC$ , et per AB dividendo  $AB - 2BP = BC$ . Capiatur  $PQ = BP$ , vt fit  $CQ = BC$ , eritque  $AQ = BC$  seu  $AQ = CQ$ , vnde angulus BQC, cui aequalis est ABC, duplus est anguli A, quae est demonstratio propositionis inuersae.

## Problema 2.

Tab. I.  
fig. 3.

5. Si in triangulo ABC angulus ABC fuerit triplus anguli A, relationem, quae hinc in latera trianguli redundat,  $AB = c$ ,  $AC = b$  et  $BC = a$  definire.

## Solutio.

Ex angulo B recta Bc ita ducatur, vt angulus CBc aequalis sit angulo A, ideoque angulus ABc eius duplus, sicque triangulum ABc ad casum praecedentis problematis pertineat. At triangulum BCc simile est triangulo ACB, vnde fit

$$AC:BC = AB:Bc = BC:Cc$$

$$b : a = c : \frac{ac}{b} = a : \frac{aa}{b}.$$

Ergo

Ergo  $Bc = \frac{ac}{b}$ , et  $Cc = \frac{aa}{b}$ , hincque  $Ac = \frac{bb - aa}{b}$ .

Iam in triangulo  $ABc$  ad analogiam ponantur latera

$$AB = \gamma, Ac = \beta, \text{ et } Bc = \alpha$$

et ex problemate praecedente habetur pro hoc triangulo ista proprietas:

$$\beta\beta - \alpha\alpha - \alpha\gamma = 0.$$

Ex modo inuentis autem nouimus esse

$$\gamma = c; \beta = \frac{bb - aa}{b}; \text{ et } \alpha = \frac{ac}{b}$$

qui valores in illa aequatione substituti praebent:

$$\frac{(bb - aa)^2}{b^2} - \frac{aac}{b} - \frac{acc}{b} = 0, \text{ siue}$$

$$(bb - aa)^2 - acc(a + b) = 0$$

quae aequatio per  $a + b$  diuisa abit in hanc:

$$(bb - aa)(b - a) - acc = 0$$

qua character indolis propositae continetur, quod angulus  $ABC$  fit triplus anguli  $A$ .

### Coroll. 1.

6. Quando ergo in triangulo  $ABC$  angulus ad  $B$  triplus est anguli  $A$ , tum inter eius latera  $AB = c$ ,  $AC = b$  et  $BC = a$  haec datur relatio, vt

$$\text{fit } (bb - aa)(b - a) - acc = 0, \text{ seu } (b - a)^2(b + a) - acc = 0,$$

quae euoluta fit

$$b^3 - abb - aab + a^3 - acc = 0.$$

### Coroll. 2.

## Coroll. 2.

7. Ad hanc proprietatem geometricè enuncian-  
dam centro C radio  $CB=a$  describatur circulus,  
latus AC productum secans in D et E, latus vero  
AB in F. Iam cum fit  $AD=b+a$ , et  $AE=b-a$ ,  
erit  $AD.AE^2=BC.AB^2$ . Ex elementis vero est  
 $AE.AD=AF.AB$ , vnde fit  $AE.AF=BC.AB$ ,  
ideoque  $AE:CE=AB:AF$ , quam proportionem  
geometricè demonstrari oportet.

## Coroll. 3.

8. In eadem figura cum fit angulus  $CFB$   
 $=ABC=3A$ , erit angulus  $ACF=2A$ , et  
 $BCD=ABC+A=4A$ , vnde arcus BD est du-  
plus arcus EF. Ducta ergo recta BE, erit ang.  
 $EBF=\frac{1}{2}ECF=A$ , ideoque  $BE=AE$ . Simili modo  
ducta recta DF, angulus ADF quoque aequatur  
angulo A, ex quo fit  $DF=AF$ .

## Coroll. 4.

9. Hinc analogia ante inuenta  $AE:CE=AB:AF$   
abit in istam  $BE:CE=AB:DF=DF+BF:DF$   
seu  $BE.DF=CE.AB=BC(BF+DF)$ . Quae  
proprietas geometricè ita ostenditur: Sumto arcu  
 $EG=EF$ , ductisque AG et BG, erit  $AG=AF$   
et  $BG=DF$ , ob arcum  $FG=BD$ , ideoque  $BFG$   
 $=BDF$  et  $AG=BG$ , ob  $AF=DF$ . Nunc vero  
ambo triangula isoscelia AGB et BCE sunt simi-  
lia, quia ang.  $CEB=2A=BAG$ ; vnde sequitur:  
 $AB:$

$AB:AG=BE:CE$ , seu  $AB:AF=AE:BC$ , vel  
 $BC.AB=AE.AF$ , quae est proprietas supra eruta.

### Scholion.

10. Inuenta ergo proprietas concinnius hoc modo geometrice demonstrabitur:

Centro C radioque CB descripto circulo latus AC in D et E, latus vero AB in F secante, ductisque BE et CF, ob angulum  $CFB=CBF=3A$ , erit angulus  $CEB=2A=CBE$ , et quia angulus  $ABE=A$ , erit  $BE=AE$ . Tum sumto arcu  $EG=EF$ , ductisque AG et BG, erit utique  $AG=AF$ , et tam  $BAG=2A$ , quam  $ABG=2A$ , ideoque  $BG=AG=AF$ . Simile ergo erit triangulum AGB triangulo BEC, vnde fit  $AB:AG=BE:BC$ , et quia  $AG=AF$ , et  $BE=AE$ , erit  $AB:AF=AE:BC$ . Ex elementis vero est  $AF:AE=AD:AB$ , vnde fit componendo

$AB:AE=AE.AD:BC.AB$ , seu  $AE^2.AD=BC.AB^2$ , quae aequatio dat  $(AC-BC)^2(AC+BC)=BC.AB^2$ , quae est proprietas supra inuenta, et nunc geometrice demonstrata.

### Problema 3.

11. Si in triangulo ABC angulus ABC fue- Tab. I.  
 rit quadruplus anguli A, inter eius latera  $AB=c$ , Fig. 4.  
 $AC=b$  et  $BC=a$ , relationem illa conditione de-  
 terminatam inuestigare.

Tom. XI. Nou. Comm.

K

Solutio.



## Solutio.

Ex angulo quadruplo B ducatur recta Bc abscindens angulum CBc = A, vt in triangulo ABc angulus ad B triplus fit anguli A, hocque triangulum ad casum problematis praecedentis pertineat. Triangulum autem BCc simile erit triangulo ACB, vnde colligitur vt ante:

$$Bc = \frac{ac}{b}, Cc = \frac{aa}{b}, \text{ hincque } Bc = \frac{bb - aa}{b}.$$

Ponantur iam pro triangulo ABc latera AB =  $\gamma$ , Ac =  $\beta$  et Bc =  $\alpha$ , et inter haec latera per problema praecedens haec relatio intercedet, vt fit:

$$\beta^2 - \alpha\beta\beta - \alpha\alpha\beta - \alpha(\gamma\gamma - \alpha\alpha) = 0.$$

Hic igitur loco  $\alpha$ ,  $\beta$ ,  $\gamma$  valores illi  $\frac{ac}{b}$ ,  $\frac{bb - aa}{b}$  et  $c$  substituantur, seu ad fractiones tollendas, quia ibi dimensionum numerus vbique est idem, hi valores per  $b$  multiplicati, quasi esset  $\alpha = ac$ ,  $\beta = bb - aa$  et  $\gamma = bc$ , scribantur; ficque exorietur haec aequatio:

$(bb - aa)^2 - ac(bb - aa)^2 - aacc(bb - aa) - ac^3(bb - aa) = 0$   
 quae cum manifesto diuiforem habeat  $bb - aa$ , erit aequatio relationem quaesitam exprimens:

$$(bb - aa)^2 - ac(bb - aa) - aacc - ac^3 = 0.$$

## C o r o l l.

12. Aequatio haec euoluta, et secundum potestates ipsius  $b$  disposita, abit in hanc formam:

$$b^4 - a(2a + c)bb - a(ac - aa)(a + c) = 0$$

qua deinceps erit vtendum.

Pro-

Problema 4.

13. Si in triangulo ABC angulus ABC fuerit quintuplus anguli A, inter eius latera  $AB=c$ ,  $AC=b$  et  $BC=a$  relationem ista conditione determinatam inuestigare.

Solutio.

Ducta iterum recta Bc, angulum CBc ipsi A aequalem abscindente, vt triangulum BCc toti ACB simile fiat, triangulum vero ABc ad casum praecedentem sit referendum, pro quo si ponamus latera  $AB=\gamma$ ,  $Ac=\beta$  et  $Bc=a$ , erit vti modo inuenimus:

$$\beta^4 - a(2a + \gamma)\beta\beta - a(\gamma\gamma - aa)(a + \gamma) = 0$$

At vero hic, vti ante ostendimus, has substitutiones fieri oportet:  $a=ac$ ,  $\beta=bb-aa$ , et  $\gamma=bc$ , vnde oritur haec aequatio:

$$(bb-aa)^4 - acc(2a+b)(bb-aa)^2 - ac^4(bb-aa)(a+b) = 0$$

quae diuisa per  $(bb-aa)(b+a)$  induit hanc formam:

$$(bb-aa)^2(b-a) - acc(2a+b)(b-a) - ac^4 = 0$$

et facta euolutione prodit

$$b^5 - ab^4 - 2aab^3 - a(cc-2aa)bb-aa(cc-aa)b - a(cc-aa)^2 = 0.$$

Problema 5.

14. Si in triangulo ABC angulus ABC fuerit sextuplus anguli A, inter eius latera  $AB=c$ ,  $AC=b$  et  $BC=a$  relationem ista conditione determinatam inuestigare.

K 2

Solutio.

## Solutio.

Ex superioribus satis iam est perspicuum, hanc relationem inueniri, si in ea, quam modo sumus adepti, loco litterarum  $a, b, c$  scribamus has formulas:  $ac, bb-aa$ , et  $bc$ ; ficque. prodit:

$$(bb-aa)^5 - ac(bb-aa)^4 - 2aacc(bb-aa)^3 - ac^3(bb-2aa)(bb-aa)^2 - aac^4(bb-aa) - ac^5(bb-aa) = 0$$

quae aequatio per  $(bb-aa)^2$  diuisa induit hanc formam:

$$(bb-aa)^3 - ac(bb-aa)^2 - 2aacc(bb-aa) - ac^3(bb-2aa) - aac^4 - ac^5 = 0$$

euolutione autem facta obtinet

$$b^6 - a(3a+c)b^4 + a(3a^2+2aac-2acc-c^2)bb - a(cc-aa)^2 (c+a) = 0 \text{ seu}$$

$$b^6 - a(c+3a)b^4 - a(c+a)(cc+ac-3aa)bb - a(cc-aa)^2 (c+a) = 0$$

## Coroll. I.

15. Si hic simili modo fiat substitutio  $a=ac$ ,  $b=bb-aa$  et  $c=bc$ , oritur aequatio pro triangulo ABC, in quo angulus ad B est septuplus anguli A, quae ergo erit:

$$(bb-aa)^6 - acc(b+3a)(bb-aa)^4 - ac^4(b+a)(bb+ab-3aa)(bb-aa)^2 - ac^6(bb-aa)^2(b+a) = 0$$

quae iam per  $(bb-aa)^2(b+a)$  diuisionem admittit, et dat

$$(bb-aa)^4(b-a) - acc(b+3a)(bb-aa)(b-a) - ac^4(bb+ab-3aa) - ac^6 = 0$$

seu

sen

$$b^7 - ab^6 - 3aab^5 - a(cc - 3aa)b^4 - aa(2cc - 3aa)b^3 - a(cc - aa)(cc - 3aa)bb - aa(cc - aa)^2b - a(cc - aa)^3 = 0.$$

## Coroll. 2.

16. Ope eiusdem substitutionis hinc obtinetur æquatio pro triangulo ABC, in quo angulus ad B est octuplus anguli A; scilicet:

$$(bb - aa)^7 - ac(bb - aa)^6 - 3aac(bb - aa)^5 - ac^2(bb - 3aa)(bb - aa)^4 - aac^2(2bb - 3aa)(bb - aa)^3 - ac^3(bb - aa)(bb - 3aa)(bb - aa)^2 - aac^3(bb - aa)^2(bb - aa) - ac^4(bb - aa)^2 = 0.$$

quæ æquatio per  $(bb - aa)^2$  diuisa præbet:

$$(bb - aa)^4 - ac(bb - aa)^3 - 3aac(bb - aa)^2 - ac^2(bb - 3aa)(bb - aa) - aac^2(2bb - 3aa) - ac^3(bb - 3aa) - aac^3 - ac^4 = 0.$$

ex cuius enolutione nascitur hæc forma:

$$b^4 - a(c + 4a)b^3 - a(c^2 + 3acc - 3aac - 6a^2)b^2 - a(c + a)(cc - aa)(cc + ac - 4aa)b - a(c + a)(cc - aa)^2 = 0.$$

## Scholion.

17. Nunc igitur rem in genere considerando, si angulus ABC ad angulum A teneat rationem  $= n : 1$ , ut sit  $ABC = n.BAC$ , positis lateribus  $AB = c$ ,  $AC = b$ , et  $BC = a$ , contemplemur æquationes pro casibus simplicioribus hætenus inuentas, quas propterea ita ordine exhibeamus, litterisque maiusculis designemus:

fi	erit
$n=1$	$b-a=0 \dots A$
$n=2$	$bb-a(a+c)=0 \dots B$
$n=3$	$b^3-abb-aab \cdot a(cc-aa)=0 \dots C$
$n=4$	$b^4-a(c+2a)bb-a(c+a)(cc-aa)=0 \dots D$
$n=5$	$b^5-ab^4-2aab^3-a(cc-2aa)bb-aa(cc-aa)b-a(cc-aa)^2=0 \dots E$
$n=6$	$b^6-a(c+3a)b^4-a(c+a)(cc+ac-3aa)bb-a(c+a)(cc-aa)^2=0 \dots F$
$n=7$	$b^7-ab^6-3aab^5-a(cc-3aa)b^4-aa(2cc-3aa)b^3-a(cc-aa)(cc-3aa)bb$ $-aa(cc-aa)^2b-a(cc-aa)^3=0 \dots G$
$n=8$	$b^8-a(c+4a)b^6-a(c^2+3acc-3aac-ba^2)b^4-a(c+a)(cc-aa)(cc+ac$ $-4aa)b^2-a(c+a)(cc-aa)^3=0 \dots H.$

Hic igitur statim constat, has formulas nonnifi alternatim sumtas commodè inter se comparari posse; quandoquidem in iis, quae numeris paribus respondent, littera  $b$  tantum pares habet dimensiones, in imparibus autem eiusdem litterae  $b$ , praeter pares, etiam impares dimensiones occurrunt, dum contra hoc casu littera  $c$  tantum pares dimensiones obtinet. Hinc istas formulas, prouti  $n$  est numerus vel par vel impar, seorsim percurramus, in legem progressionis inquiraturi, ubi quidem primo ostendam, utroque casu has formulas seriem recurrentem constituere, cuius quisque terminus per binos praecedentes determinatur; deinde vero etiam formam generalem exhibere conabor.

### Problema 6.

18. Si in triangulo ABC fuerit angulus  $B=2iA$ , denotante  $2i$  numerum integrum parèm quem-

quemcunque, naturam relationis, quae inter trian-  
guli latera  $AB=c$ ,  $AC=b$  et  $BC=a$  intercedit,  
inuestigare.

## Solutio.

Hic igitur considerari oportet progressionem  
earum alternarum formularum, quas ante litteris  
B, D, F, H etc. designauimus, quae ita se habent:

fi	inuenta est formula
$i=1$	$B=bb-a(c+a)=0$
$i=2$	$D=b^4-a(c+2a)bb-a(c+a)(cc-aa)=0$
$i=3$	$F=b^6-a(c+3a)b^4-a(c+a)(cc+ac-3aa)bb$ $-a(c+a)(cc-aa)^2=0$
$i=4$	$H=b^8-a(c+4a)b^6-a(c^2+3acc-3aac-6a^3)b^4$ $-a(c+a)(cc-aa)(cc+ac-4aa)bb-a(c+a)(cc-aa)^3=0$

quarum formularum lex quo facilius obseruetur, eas  
etiam secundum potestates litterae  $c$  disponamus:

fi	erit
$i=1$	$B=(bb-aa)-ac=0$
$i=2$	$D=(bb-aa)^2-ac(bb-aa)-aac-aa^2=0$
$i=3$	$F=(bb-aa)^3-ac(bb-aa)^2-2aac(bb-aa)$ $-ac^2(bb-2aa)-aac^2-ac^3=0$
$i=4$	$H=(bb-aa)^4-ac(bb-aa)^3-3aac(bb-aa)^2$ $-ac^2(bb-3aa)(bb-aa)-aac^2(2bb-3aa)$ $-ac^3(bb-3aa)-aac^3-ac^4=0$

Hic primum obseruo, si a quavis formula praecedens  
per  $bb-aa$  multiplicata subtrahatur, residua multo  
simpliciora esse proditura; erit enim:

D-B

$$D - B(bb - aa) = -aacc - ac^2$$

$$F - D(bb - aa) = -aacc(bb - aa) + a^2c^2 - aac^2 - ac^2$$

$$H - F(bb - aa) = -aacc(bb - aa)^2 + a^2c^2(bb - aa) - aac^2(bb - 2aa) + 2a^2c^2 - aac^2 - aa^2$$

subtrahatur insuper a qualibet praecedens per  $cc$  multiplicata, reperieturque:

$$D - B(bb - aa + cc) = -bbcc$$

$$F - D(bb - aa + cc) = -bbcc(bb - aa) + abbc^2$$

$$H - F(bb - aa + cc) = -bbcc(bb - aa)^2 + abbc^2(bb - aa) + aabbc^2 + abbc^2$$

quae formae per  $-bbcc$  diuisae praebent

$$\frac{B(bb - aa + cc) - D}{bbcc} = I$$

$$\frac{D(bb - aa + cc) - F}{bbcc} = bb - aa - ac = B$$

$$\frac{F(bb - aa + cc) - H}{bbcc} = (bb - aa)^2 - ac(bb - aa) - aacc - ac^2 = D$$

vbi profecto casu non euenire videtur, vt primo quidem vnitas, tum vero ipsae litterae B et D prodeant; pro inductione quidem hinc stabilienda hi duo casus certe minime sufficerent, verum calculo ad sequentem formulam K continuato, non solum idem contingit, sed etiam pro formulis ordine imparibus A, C, E, G deinceps eadem lex progressionis deprehendetur. Quam ob rem non dubito, huic inductioni innixus pronunciare, formulas has B, D, F, H etc. seriem constituere recurrentem, cuius scala relationis  $bb - aa - cc$ ,  $-bbcc$ , hincque terminum antecedentem ipsi  $i = 0$  respondentem esse vnitatem. Ita his formulis ita dispositis:

$$i \dots 0, 1, 2, 3, 4, 5, 6, 7 \\ I, B, D, F, H, K, M, O \text{ etc.}$$

erit

erit primo quidem  $B = bb - aa - ac$ , tum vero secundum legem seriei recurrentis:

$$\begin{aligned} D &= (bb - aa + cc)B - bbcc. \text{ pro } n=4 \\ F &= (bb - aa + cc)D - bbccB \text{ pro } n=6 \\ H &= (bb - aa + cc)F - bbccD \text{ pro } n=8 \\ K &= (bb - aa + cc)H - bbccF \text{ pro } n=10 \\ M &= (bb - aa + cc)K - bbccH \text{ pro } n=12 \\ &\text{etc.} \end{aligned}$$

vnde has formulas, quousque lubuerit, continuare licet.

### Coroll. 1.

19. Si ergo haec series formetur:  $1 + Pz + Dz^2 + Fz^3 + \text{etc.}$  ea ex evolutione huiusmodi fractionis:

$$\frac{1 + \Delta z}{1 - (bb + aa + cc)z + bbcczz}$$

nascitur, vbi quidem est  $\Delta = -ac - cc$ , haecque fractio adeo illius seriei in infinitum prolatae summam exhibet.

### Coroll. 2.

20. Hinc porro in genere formulam indefinite numero  $i$  conuenientem exhibere licet, quippe quae ita exprimetur:

$$\begin{aligned} \mathfrak{A} &\left( \frac{bb - aa + cc + \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}{2} \right)^i \\ \mathfrak{B} &\left( \frac{bb - aa + cc - \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}}{2} \right)^i \end{aligned}$$



vbi quidem, applicatione ad duas primores facta, fit

$$\mathcal{A} + \mathcal{B} = 1$$

$$\text{et } \frac{bb - aa + cc}{2} + \frac{1}{2}(\mathcal{A} - \mathcal{B})\sqrt{\dots} = bb - aa - ac$$

$$\text{seu } \mathcal{A} - \mathcal{B} = \frac{bb - (a+c)^2}{\sqrt{\dots}} = -\sqrt{\frac{(a+c+b)(a+c-b)}{(a+b-c)(b+c-a)}}$$

### Scholiom.

Tab. I. 21. Formula haec generalis eo maiori cura

Fig. 5. euolui meretur, quod adhuc soli inductioni innititur, ideoque vberiori confirmatione indiget. Sint

igitur trianguli ABC latera  $AB=c$ ,  $AC=b$ ,  $BC=a$ , et anguli  $A=\alpha$ ,  $B=\beta$ ,  $C=\gamma$ , vbi quidem assumimus esse  $\beta=2i\alpha$ . Nunc vero est

$$\cos \alpha = \frac{bb - aa + cc}{2bc} \text{ et fin. } \alpha = \frac{\sqrt{2aabb + 2aaccc + 2bbcc - a^4 - b^4 - c^4}}{2bc}$$

ex quo formula nostra inuenta induet hanc formam:

$$\mathcal{A}(bc \cos \alpha + b^2 - 1 \sin \alpha)^2 + \mathcal{B}(bc \cos \alpha - bc\sqrt{-1} \sin \alpha)^2$$

quae per principia nota transfunditur in hanc:

$$\mathcal{A}b^2i^2(\cos i\alpha + \sqrt{-1} \sin i\alpha) + \mathcal{B}b^2i^2(\cos i\alpha - \sqrt{-1} \sin i\alpha)$$

Cum vero sit  $\mathcal{A} + \mathcal{B} = 1$ , et  $\mathcal{A} - \mathcal{B} = \frac{bb - (a+c)^2}{2bc\sqrt{-1} \sin \alpha}$ ,

ex formula  $\cos \beta = \cos 2i\alpha = \frac{aa + bc - bb}{2ac}$ , sequitur fore

$$1 + \cos 2i\alpha = \frac{(a+c)^2 - bb}{2ac}, \text{ ideoque } \mathcal{B} - \mathcal{A} = \frac{a(1 + \cos 2i\alpha)}{b\sqrt{-1} \sin \alpha}$$

at  $\sin \alpha : \sin 2i\alpha = a : b$ , vnde  $\mathcal{B} - \mathcal{A} = \frac{1 + \cos 2i\alpha}{\sqrt{-1} \sin 2i\alpha}$ , seu

$\mathcal{B} - \mathcal{A} = \frac{\cos i\alpha}{\sqrt{-1} \sin i\alpha}$ . Quo circa habebitur:

$$\mathcal{A} = \frac{-\cos i\alpha + \sqrt{-1} \sin i\alpha}{2\sqrt{-1} \sin i\alpha}, \text{ et } \mathcal{B} = \frac{\cos i\alpha + \sqrt{-1} \sin i\alpha}{2\sqrt{-1} \sin i\alpha}$$

sicque

Etque formula inuenta fit

$$\frac{b^i c^i}{2V - 1. \sin. i\alpha} ((\cos. i\alpha + V - 1. \sin. i\alpha)(-\cos. i\alpha + V - 1. \sin. i\alpha) + (\cos. i\alpha - V - 1. \sin. i\alpha)(\cos. i\alpha + V - 1. \sin. i\alpha))$$

quae cum sponte in nihilum abeat, euidentis est, casu, quo angulus  $\beta = 2i\alpha$ , formulam inuentam nihilo esse aequalem, ideoque inductionem veritati consentaneam.

### Problema 7.

22. Si in triangulo ABC fuerit angulus  $\beta = (2i + 1)\alpha$ , existente  $2i + 1$  numero impari quocunque integro, naturam relationis, quae hinc inter latera trianguli  $a, b, c$  intercedit, inuestigare.

### Solutio.

Ex serie ergo formularum supra (17) exhibitae, eas alternas considerari oportet, quae litteris A, C, E, G etc. sunt designatae, et ordine expositae, ita se habent:

fi	formula inuenta est
$i = 0$	$A = b - a = 0$
$i = 1$	$C = b^3 - abb - aab - a(cc - aa) = 0$
$i = 2$	$E = b^5 - ab^4 - 2aab^3 - a(cc - 2aa)bb - aa(cc - aa)b - a(cc - aa)^2 = 0$
$i = 3$	$G = b^7 - ab^6 - 3aab^5 - a(cc - 3aa)b^4 - aa(2cc - 3aa)b^3 - a(cc - aa)(cc - 3aa)bb - aa(cc - aa)^2b - a(cc - aa)^3 = 0$

L 2

quae

quae eadem secundum potestates ipsius  $c$  dispositae ita repraesententur :

fi	erit
$i=0$	$A=(b-a)=0$
$i=1$	$C=(b-a)(bb-aa)-acc=0$
$i=2$	$E=(b-a)bb-aa)^2-acc(bb+ab-2aa)-ac^2=0$
$i=3$	$G=(b-a)(bb-aa)^3-acc(bb-aa)(bb+2ab-3aa)-ac^2(bb+ab-3aa)-ac^3=0$

Atque ex his colligimus primo :

$$(bb-aa)A-C=acc$$

$$(bb-aa)C-E=acc(b-a)+ac^2$$

$$(bb-aa)E-G=acc(bb-aa)(b-a)+acc^2(b-2a)+ac^3$$

tum vero porro :

$$(bb-aa+cc)A-C=bcc$$

$$(bb-aa+cc)C-E=bbcc(b-a)=bbccA$$

$$(bb-aa+cc)E-G=bbcc(b-a)(bb-aa)-abbc^2=bbccC.$$

Vnde iam multo maiori fiducia concludimus, has formulas A, C, E, G etc. seriem recurrentem constituisse, cuius scala relationis sit  $bb-aa+cc$ , et terminum primo A praecedentem censendum esse  $=\frac{1}{b}$ . Quare ex cognitis duobus primis  $A=b-a$  et  $C=(b-a)(bb-aa)-acc$  sequentes hac lege formantur.

$$E=(bb-aa+cc)C-bbccA \text{ pro } n=5$$

$$G=(bb-aa+cc)E-bbccC \text{ pro } n=7$$

$$I=(bb-aa+cc)G-bbccE \text{ pro } n=9$$

$$L=(bb-aa+cc)I-bbccG \text{ pro } n=11$$

$$N=(bb-aa+cc)L-bbccI \text{ pro } n=13$$

etc.

Coroll.

Coroll. i.

23. His igitur formulis ita secundum numeros  $i$  dispositis, erit

$i \dots 0, 1, 2, 3, 4, 5, 6$   
formula A, C, E, G, I, L, N etc.  
et terminus indefinite numero  $i$  conueniens erit, vt  
ante, huius formae:

$$\mathfrak{A} \left( \frac{bb - aa + cc + \sqrt{(a^2 + b^2 + c^2 - 2aabb - 2aacc - 2bbcc)}}{2} \right)^i \\ + \mathfrak{B} \left( \frac{bb - aa + cc - \sqrt{(a^2 + b^2 + c^2 - 2aabb - 2aacc - 2bbcc)}}{2} \right)^i$$

Coroll. i.

24. Coefficientes  $\mathfrak{A}$  et  $\mathfrak{B}$  ex binis terminis initialibus A et C ita definiuntur, vt primo fit

$$\mathfrak{A} + \mathfrak{B} = A = b - a, \text{ tum vero } \frac{(b-a)(bb-aa+cc)}{2} + \frac{1}{2}$$

$$(\mathfrak{A} - \mathfrak{B}) \sqrt{(\dots)} = (b-a)(bb-aa) - acc$$

$$\text{ergo } (\mathfrak{A} - \mathfrak{B}) \sqrt{(\dots)} = (b-a)(bb-aa - (b+a)cc) \\ = (b+a)(bb - 2ab + aa - cc)$$

$$\text{hincque } \mathfrak{A} - \mathfrak{B} = \frac{(b+a)((b-a)^2 - cc)}{\sqrt{(a^2 + b^2 + c^2 - 2aabb - 2aacc - 2bbcc)}}$$

Scholion.

25. Euoluamus hanc formulam generalem pari modo, quo ante fecimus (21), eritque prorsus vt ante forma nostra generalis:

$$\mathfrak{A}(bcc \cos. a + bc \sqrt{-1. \sin. a})^i + \mathfrak{B}(bcc \cos. a - bc \sqrt{-1. \sin. a})^i$$

quae pariter in hanc abit:

$$\mathfrak{A} b^i c^i (\cos. ia + \sqrt{-1. \sin. ia}) + \mathfrak{B} b^i c^i (\cos. ia - \sqrt{-1. \sin. ia})$$

L 3

vbi

vbi est  $\mathcal{A} + \mathcal{B} = b - a$ , et  $\mathcal{A} - \mathcal{B} = \frac{(b+a)((b-a)^2 - cc)}{2bc\sqrt{1 - \sin. \alpha}}$ .

Nunc vero, ob ang.  $\gamma = 180^\circ - 2(i+1)\alpha$ , erit

$$\cos. 2(i+1)\alpha = \frac{cc - aa - bb}{2ab} \text{ et } 1 + \cos. 2(i+1)\alpha = \frac{cc - (b-a)^2}{2ab}$$

$$\text{vnde } \mathcal{B} - \mathcal{A} = \frac{a(b+a)(1 + \cos. 2(i+1)\alpha)}{c\sqrt{1 - \sin. \alpha}}.$$

At est  $a:c = \sin. \alpha : \sin. 2(i+1)\alpha$ , ideoque  $\mathcal{B} - \mathcal{A}$

$$= \frac{(b+a)\cos.(i+1)\alpha}{\sqrt{1 - \sin.(i+1)\alpha}}, \text{ vnde colligimus}$$

$$\mathcal{A} = \frac{(b+a)\cos.(i+1)\alpha + (b-a)\sqrt{1 - \sin.(i+1)\alpha}}{2\sqrt{1 - \sin.(i+1)\alpha}} \text{ et}$$

$$\mathcal{B} = \frac{(b+a)\cos.(i+1)\alpha + (b-a)\sqrt{1 - \sin.(i+1)\alpha}}{2\sqrt{1 - \sin.(i+1)\alpha}}.$$

Quoniam latera  $a$  et  $b$  sunt finibus angulorum oppositorum proportionalia, statuamus  $a = 2f \sin. \alpha$ , et  $b = 2f \sin. (2i+1)\alpha$ , eritque

$$a \cos.(i+1)\alpha = f(\sin.(i+2)\alpha - \sin.i\alpha); a \sin.(i+1)\alpha = f(\cos.i\alpha - \cos.(i+2)\alpha)$$

$$b \cos.(i+1)\alpha = f(\sin.(3i+2)\alpha + \sin.i\alpha); b \sin.(i+1)\alpha = f(\cos.i\alpha - \cos.(3i+2)\alpha)$$

vnde colligimus:

$$(a+b) \cos.(i+1)\alpha = f(\sin.(i+2)\alpha + \sin.(3i+2)\alpha) \text{ et}$$

$$(b-a) \sin.(i+1)\alpha = f(\cos.(i+2)\alpha - \cos.(3i+2)\alpha).$$

Quare cum fit generatim

$$\sin. \mu + \sin. \nu = 2 \sin. \frac{\mu + \nu}{2} \cos. \frac{\nu - \mu}{2} \text{ et}$$

$$\cos. \mu - \cos. \nu = 2 \sin. \frac{\mu + \nu}{2} \sin. \frac{\nu - \mu}{2}$$

habebimus;

$$(a+b) \cos.(i+1)\alpha = 2f \sin.(2i+2)\alpha \cos.i\alpha \text{ et}$$

$$(b-a) \sin.(i+1)\alpha = 2f \sin.(2i+2)\alpha \sin.i\alpha$$

ac propterea adipiscimur:

$$\mathcal{A} = \frac{f \sin. (2i+2) \alpha}{\sqrt{-1} \sin. (i+1) \alpha} (-\cos. i \alpha + \sqrt{-1} \sin. i \alpha) \text{ et}$$

$$\mathcal{B} = \frac{f \sin. (2i+2) \alpha}{\sqrt{-1} \sin. (i+1) \alpha} (\cos. i \alpha + \sqrt{-1} \sin. i \alpha)$$

unde perspicuum est, fore

$$\mathcal{A}(\cos. i \alpha + \sqrt{-1} \sin. i \alpha) + \mathcal{B}(\cos. i \alpha - \sqrt{-1} \sin. i \alpha) = 0,$$

quo ipso veritas inductionis nostrae euincitur. His autem obseruatis, nunc demum solutionem nostri Problematis directe aggredi licet.

### Problema 8.

26. Si in triangulo ABC angulus B ad angulum A rationem teneat quamcunque multiplam, ut  $n$  ad 1, relationem, quae inde inter latera trianguli  $AB=c$ ,  $AC=b$ ,  $BC=a$  intercedit, analytice inuestigare.

### Solutio.

Posito angulo  $A=\alpha$ , ut sit angulus  $B=n\alpha$ , erit, vti ex angulorum doctrina constat:

$$\cos. n\alpha + \sqrt{-1} \sin. n\alpha = (\cos. \alpha + \sqrt{-1} \sin. \alpha)^n \text{ et}$$

$$\cos. n\alpha - \sqrt{-1} \sin. n\alpha = (\cos. \alpha - \sqrt{-1} \sin. \alpha)^n, \text{ ideoque}$$

$$\frac{\cos. n\alpha + \sqrt{-1} \sin. n\alpha}{\cos. n\alpha - \sqrt{-1} \sin. n\alpha} = \left( \frac{\cos. \alpha + \sqrt{-1} \sin. \alpha}{\cos. \alpha - \sqrt{-1} \sin. \alpha} \right)^n.$$

Iam prout  $n$  est numerus par vel impar, duo casus sunt euoluendi

Sit primo  $n=2i$ , et vtrinque radix quadrata extrahatur, fietque:

$$\frac{\cos. i\alpha + \sqrt{-1} \sin. i\alpha}{\cos. i\alpha - \sqrt{-1} \sin. i\alpha} = \left( \frac{\cos. \alpha + \sqrt{-1} \sin. \alpha}{\cos. \alpha - \sqrt{-1} \sin. \alpha} \right)^i$$

quod

nunc vero est  $\cos. 2i\alpha = \frac{aa+cc-bb}{2ac}$ , ideoque

$$\cos. i\alpha = \frac{1}{2} \sqrt{\frac{(a+c)^2 - bb}{ac}} \text{ et } \sin. i\alpha = \frac{1}{2} \sqrt{\frac{bb - (c-a)^2}{ac}}$$

$$\text{deinde } \cos. \alpha = \frac{bb-aa+cc}{2bc} \text{ et } \sin. \alpha = \frac{\sqrt{(2aabb+2amcc+2bbcc-a^4-b^4-c^4)}}{2bc}$$

Sit breuitatis gratia

$$\Delta = \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)}$$

et nostra aequatio fit

$$\frac{(a+c)^2 - bb + \Delta}{(a+c)^2 - bb - \Delta} = \left( \frac{bb - aa + cc + \Delta}{bb - aa + cc - \Delta} \right)^i \text{ seu}$$

$$((a+c)^2 - bb - \Delta)(bb - aa + cc + \Delta)^i - ((a+c)^2 - bb + \Delta)(bb - aa + cc - \Delta)^i = 0$$

quae per  $2\Delta$  diuisa conuenit cum forma supra inuenta. Sit deinde  $n = 2i + 1$ , et multiplicando

aequationem per  $\frac{\cos. \alpha - \sqrt{-1} \sin. \alpha}{\cos. \alpha + \sqrt{-1} \sin. \alpha}$  oriatur,

$$\frac{\cos. 2i\alpha - \sqrt{-1} \sin. 2i\alpha}{\cos. 2i\alpha + \sqrt{-1} \sin. 2i\alpha} = \left( \frac{\cos. \alpha + \sqrt{-1} \sin. \alpha}{\cos. \alpha - \sqrt{-1} \sin. \alpha} \right)^{2i}$$

et quadratam radicem extrahendo:

$$\frac{\cos. i\alpha + \sqrt{-1} \sin. i\alpha}{\cos. i\alpha - \sqrt{-1} \sin. i\alpha} = \left( \frac{\cos. \alpha + \sqrt{-1} \sin. \alpha}{\cos. \alpha - \sqrt{-1} \sin. \alpha} \right)^i$$

Cum nunc sit  $\gamma = 180^\circ - 2(i+1)\alpha$ , erit

$$\cos. 2(i+1)\alpha = \frac{cc - bb - aa}{2ab} \text{ hincque}$$

$$\cos. (i+1)\alpha = \frac{1}{2} \sqrt{\frac{cc - (b-a)^2}{ab}} \text{ et } \sin. (i+1)\alpha = \frac{1}{2} \sqrt{\frac{(b+a)^2 - cc}{ab}}$$

$$\text{Est vero } \cos. \alpha = \frac{bb - aa + cc}{2bc} \text{ et } \sin. \alpha \sqrt{-1} = \frac{\Delta}{2bc} \text{ seu } \sin. \alpha = \frac{\Delta}{2bc\sqrt{-1}},$$

vbi notatur esse  $\frac{1}{\sqrt{-1}} = \sqrt{(bc - (b-a)^2)((b+a)^2 - cc)}$ ; quamobrem elicietur

$$\cos. i\alpha = \frac{1}{2bc\sqrt{ab}} ((bb - aa + cc)\sqrt{(cc - (b-a)^2)} + ((b+a)^2 - cc)\sqrt{(cc - (b-a)^2)})$$

seu

feu  $\cos.ia = \frac{b+a}{2c\sqrt{ab}} \sqrt{cc-(b-a)^2}$ , tum vero

$$\sin.ia = \frac{1}{b\sqrt{ab}} ((bb-aa+cc) \sqrt{(b+a)^2-cc} - (cc-(b-a)^2) \sqrt{(b+a)^2-cc})$$

feu  $\sin.ia = \frac{b-a}{2c\sqrt{ab}} \sqrt{(b+a)^2-cc}$ . Quibus substitutis erit:

$$\frac{(b+a)(cc-(b-a)^2) + (b-a)\Delta}{(b+a)(cc-(b-a)^2) - (b-a)\Delta} = \frac{(bb-aa+cc+\Delta)^i}{(bb-aa+cc-\Delta)^i}$$

et aequatio hinc supra inuenta colligitur:

$$(b+a - \frac{(b-a)\Delta}{cc-(b-a)^2}) (bb-aa+cc+\Delta)^i - (b+a + \frac{(b-a)\Delta}{cc-(b-a)^2}) (bb-aa+cc-\Delta)^i = 0$$

dummodo haec ducatur in  $\frac{cc-(b-a)^2}{2\Delta}$ , atque ex hac forma simul natura seriei recurrentis intelligitur.

## Coroll. I.

27. Pro casu ergo quo in triangulo ABC angulus  $B = 2iA$  aequatio laterum relationem exprimens est:

$$(1 + \frac{bb-(a+c)^2}{\Delta}) (bb-aa+cc+\Delta)^i + (1 - \frac{bb-(a+c)^2}{\Delta}) (bb-aa+cc-\Delta)^i = 0$$

pro casu autem, quo angulus  $B = (2i+1)A$ , habetur:

$$(b-a - \frac{(b+a)(cc-(b-a)^2)}{\Delta}) (bb-aa+cc+\Delta)^i + (b-a + \frac{(b+a)(cc-(b-a)^2)}{\Delta}) (bb-aa+cc-\Delta)^i = 0.$$



## Coroll. 2.

28. Quodsi ergo has constituamus formas:

$$\frac{1}{2}(\frac{bb-aa+cc+\Delta}{2})^i + \frac{1}{2}(\frac{bb-aa+cc-\Delta}{2})^i = V$$

$$\frac{1}{2\Delta}(\frac{bb-aa+cc+\Delta}{2})^i - \frac{1}{2\Delta}(\frac{bb-aa+cc-\Delta}{2})^i = W$$

quarum vtraque est rationalis non obstante formula irrationali:

$$\begin{aligned}\Delta &= \sqrt{(a^4 + b^4 + c^4 - 2aabb - 2aacc - 2bbcc)} \\ &= \sqrt{((bb-aa+cc)^2 - 4bbcc)}\end{aligned}$$

pro casu  $B=2iA$  erit

$$V + (bb - (a+c)^2)W = 0$$

pro casu vero  $B=(2i+1)A$  erit

$$(b-a)V + (b+a)((b-a)^2 - cc)W = 0$$

## Coroll. 3.

29. Quodsi pro singulis valoribus numeri integri  $i$  ambae formae  $V$  et  $W$  euoluantur, binae exorientur series recurrentes per eandem scalam relationis  $bb-aa+cc$ ,  $-bbcc$  continuandae, ex quibus deinceps ambae illae triangulorum proprietates facile exhibentur.

## Scholion.

30. Quo has series succinctius exprimamus fit breuitatis gratia  $bb-aa+cc=ff$ , et pro serie priori  $V = (\frac{ff+\Delta}{2})^i + (\frac{ff-\Delta}{2})^i$

Ob

Ob  $\Delta = V(f^4 - 4bbcc)$  et scalam relationis  $ff, -bbcc$  inueniemus:

fi	valores ipsius V
$i=0$	2
$i=1$	$ff$
$i=2$	$f^4 - 2bbcc$
$i=3$	$f^6 - 3bbccff$
$i=4$	$f^8 - 4bbccf^2 + 2b^4c^4$
$i=5$	$f^{10} - 5bbccf^4 + 5b^4c^4ff$
$i=6$	$f^{12} - 6bbccf^6 + 9b^4c^4f^2 - 2b^6c^6$
$i=7$	$f^{14} - 7bbccf^8 + 14b^4c^4f^4 - 7b^6c^6f^2$

vnde generatim colligitur fore  $V =$

$$f^{2i} - i b b c c f^{2i-2} + \frac{i(i-3)}{1 \cdot 2} b^4 c^4 f^{2i-4} - \frac{i(i-4)(i-5)}{1 \cdot 2 \cdot 3} b^6 c^6 f^{2i-6} + \text{etc.}$$

Deinde pro altera forma  $W = \frac{1}{\Delta} \left( \frac{ff + \Delta}{2} \right)^i - \frac{1}{\Delta} \left( \frac{ff - \Delta}{2} \right)^i$

seguens nascetur series:

fi	valores ipsius W
$i=0$	0
$i=1$	1
$i=2$	$ff$
$i=3$	$f^4 - bbcc$
$i=4$	$f^6 - 2bbccff$
$i=5$	$f^8 - 3bbccf^2 + b^4c^4$
$i=6$	$f^{10} - 4bbccf^4 + 3b^4c^4ff$
$i=7$	$f^{12} - 5bbccf^6 + 6b^4c^4f^2 - b^6c^6$

vnde in genere haec forma erit  $W =$

$$f^{2i-2} - (i-2) b b c c f^{2i-4} + \frac{(i-3)(i-4)}{1 \cdot 2} b^4 c^4 f^{2i-6} - \frac{(i-4)(i-5)(i-6)}{1 \cdot 2 \cdot 3} b^6 c^6 f^{2i-8} + \text{etc.}$$

M 2

vbi

vbi probe notandum est, has duas expressiones generales tantum vsque ad terminos evanescentes proferri debere, etiam si deinceps denuo termini finiti redeant. Caeterum hinc patet, fore  $\frac{1}{2}(V + ffW) =$

$$f^{2i} - (i-1)bbccf^{2i-4} + \frac{(i-2)(i-3)}{1 \cdot 2} b^4 c^4 f^{2i-8} \\ - \frac{(i-5)(i-4)(i-3)}{1 \cdot 2 \cdot 3} b^6 c^6 f^{2i-12} + \text{etc.}$$

atque  $\frac{1}{2}(V - ffW) =$

$$-bbccf^{2i-4} + (i-3)b^4 c^4 f^{2i-8} - \frac{(i-4)(i-5)}{1 \cdot 2} b^6 c^6 f^{2i-12} \\ + \frac{(i-5)(i-6)(i-7)}{1 \cdot 2 \cdot 3} b^8 c^8 f^{2i-16} - \text{etc.}$$

Hinc iam pro casu triangulorum, vbi angulus  $B = 2iA$ ; ob  $bb = ff + aa - cc$ , aequatio laterum relationem exprimens erit:

$$\frac{1}{2}(V + ffW) - c(a+c)W = 0$$

pro altero autem casu, vbi angulus  $B = (2i+1)A$ , posito hic  $cc = ff - bb + aa$ , aequatio laterum relationem exprimens erit:

$$\frac{1}{2}b(V - ffW) - \frac{1}{2}a(V + ffW) + b(bb - aa)W = 0.$$

Verum etiam alio modo hae expressiones generales absque introductione quantitatis  $ff = bb - aa + cc$  repraesentari possunt, vti in sequenti problemate videbimus.

### Problema 9.

31. Si in triangulo ABC angulus B ad angulum A rationem teneat quamcunque multiplam, vt  $n$  ad 1, aequationem, qua relatio inter latera

AB

$AB=c$ ,  $AC=b$  et  $BC=a$  exprimitur, in genere exhibere.

## Solutio.

Si aequationes pro singulis casibus supra inventas attentius consideremus, haud difficulter legem certam in terminorum progressu observabimus, ex indole progressionis demonstrata facile confirmandam. Duos autem casus hic distingui oportet, prout numerus ille  $n$  fuerit par, vel impar. Pro utroque autem casu aequatio quaesita sequenti modo exhiberi poterit:

Pro casu, quo  $n=2i$ .

aequatio laterum relationem exprimens ita se habet:

$$\begin{aligned} \frac{b^{2i}}{a} = & +c b^{2i-2} + c(cc - (i-1)aa) b^{2i-4} + c(c^2 - 2(i-2)aacc + \frac{(i-2)(i-1)}{1 \cdot 2} a^4) b^{2i-6} \\ & + iab^{2i-2} + a((i-1)cc - \frac{(i-1)i}{1 \cdot 2} aa) b^{2i-4} + a(i-2)c^2 - \frac{2(i-2)(i-1)}{1 \cdot 2} a^2 c^2 + \frac{(i-2)(i-1)i}{1 \cdot 2 \cdot 3} a^4) b^{2i-6} \\ & + c(c^2 - 3(i-3)a^2 c^2 + \frac{3(i-3)(i-2)}{1 \cdot 2} a^4 c^2 - \frac{(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3} a^6) b^{2i-8} \\ & + a((i-3)c^2 - \frac{3(i-3)(i-2)}{1 \cdot 2} a^2 c^2 + \frac{3(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3} a^4 c^2 - \frac{(i-3)(i-2)(i-1)i}{1 \cdot 2 \cdot 3 \cdot 4} a^6) b^{2i-10} \\ & + c(c^2 - 4(i-4)a^2 c^2 + \frac{6(i-4)(i-3)}{1 \cdot 2} a^4 c^2 - \frac{4(i-4)(i-3)(i-2)}{1 \cdot 2 \cdot 3} a^6 c^2 + \frac{(i-4)(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3 \cdot 4} a^8) b^{2i-12} \\ & + a((i-4)c^2 - \frac{4(i-4)(i-3)}{1 \cdot 2} a^2 c^2 + \frac{6(i-4)(i-3)(i-2)}{1 \cdot 2 \cdot 3} a^4 c^2 - \frac{4(i-4)(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3 \cdot 4} a^6 c^2 \\ & + \frac{(i-4)(i-3)(i-2)(i-1)i}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^8) b^{2i-14} \end{aligned}$$

cuius lex continuationis satis est manifesta.

Pro casu, quo  $n=2i+1$ .

aequatio laterum relationem exprimens ita se habet:

$$\begin{aligned} \frac{b^{2i+1}}{a} = & +b^{2i} + (cc - iaa)b^{2i-2} + (c^4 - 2(i-1)aa'cc + \frac{(i-1)^2}{1 \cdot 2} a^4)b^{2i-4} \\ & +iab^{2i-1} + a((i-1)cc - \frac{(i-1)}{1 \cdot 2} aa')b^{2i-3} + a((i-2)c^4 - \frac{2(i-2)(i-1)}{1 \cdot 2} a^2c^2 + \frac{(i-2)(i-1)^2}{1 \cdot 2 \cdot 3} a^4)b^{2i-5} \\ & + (c^6 - 3(i-2)a^2c^4 + 3\frac{(i-2)(i-1)}{1 \cdot 2} a^4c^2 - \frac{(i-2)(i-1)^2}{1 \cdot 2 \cdot 3} a^6)b^{2i-6} \\ & + a((i-3)c^6 - 3\frac{(i-3)(i-2)}{1 \cdot 2} a^2c^4 + 3\frac{(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3} a^4c^2 - \frac{(i-3)(i-2)(i-1)^2}{1 \cdot 2 \cdot 3 \cdot 4} a^6)b^{2i-7} \\ & \text{etc.} \end{aligned}$$

quae aequatio commodius hac forma, secundum potestates ipsius  $c$  disposita, repraesentari potest:

$$\begin{aligned} \frac{b^{2i+1}}{a} = & c^{2i} + c^{2i-2} \left\{ bb - iaa \right\} + c^{2i-4} \left\{ b^4 - 2(i-1)aa'bb + \frac{(i-1)^2}{1 \cdot 2} a^4 \right\} \\ & + c^{2i-6} \left\{ b^6 - 3(i-2)a^2b^4 + \frac{3(i-2)(i-1)}{1 \cdot 2} a^4b^2 - \frac{(i-2)(i-1)^2}{1 \cdot 2 \cdot 3} a^6 \right\} \\ & + c^{2i-8} \left\{ b^8 - 4(i-3)a^2b^6 + \frac{6(i-3)(i-2)}{1 \cdot 2} a^4b^4 - \frac{4(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3} a^6b^2 + \frac{(i-3)(i-2)(i-1)^2}{1 \cdot 2 \cdot 3 \cdot 4} a^8 \right\} \\ & + c^{2i-10} \left\{ b^{10} - 5(i-4)a^2b^8 + \frac{10(i-4)(i-3)}{1 \cdot 2} a^4b^6 - \frac{10(i-4)(i-3)(i-2)}{1 \cdot 2 \cdot 3} a^6b^4 + \frac{5(i-4)(i-3)(i-2)(i-1)}{1 \cdot 2 \cdot 3 \cdot 4} a^8b^2 - \frac{(i-4)(i-3)(i-2)(i-1)^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^{10} \right\} \\ & \text{etc.} \end{aligned}$$

### Scholion.

32. His considerationibus doctrina triangulorum non mediocriter amplificari videtur, dum statim atque in quopiam triangulo ratio inter binos eius angulos innotescit, simul relatio certa inter eius latera exhiberi potest. Cum autem haec nimis sint generalia, quandoquidem ex hac relatione unicum

cum latus per bina reliqua determinatur, conueniet, has proprietates generales inuentas ad certam triangulorum speciem accommodari, vbi quidem triangula ifoscelia prae caeteris sunt notatu digna, quia in iis saepenumero ratio inter angulum verticalem et angulos ad basin praescribi solet, quoties scilicet polygona regularia sunt construenda. Duo autem casus hic euoluendi occurrunt, prout vel angulus ad basin est multiplus anguli verticalis, vel angulus verticalis multiplus anguli ad basin; quos ambos in sequentibus problematibus sum expediturus.

### Problema 10.

33. Si in triangulo ifoscele BAC angulus ad basin fuerit multiplus anguli verticalis A in ratione  $n:1$ , inuestigare relationem inter basin  $BC=a$  et latera  $AB=AC=b$ . Tab. I.  
Fig. 6.

### Solutio.

Primum obseruandum est; ob hanc rationem ipsos angulos dari; posita enim mensura duorum angulorum rectorum  $=\pi$ , et angulo verticali  $A=\alpha$ , ob  $\alpha + 2n\alpha = \pi$ , fit  $\alpha = \frac{\pi}{2n+1}$ . Iam formulis ante inuentis huc transferendis, erit  $c=b$ , et binis casibus seorsim tractatis, prout  $n$  est numerus vel par vel impar, quorum utroque formulae seriem recurrentem constituunt, cuius scala relationis est  $2bb-aa, -b^4$ , primo ponendo  $n=2i$ , habebimus:

fi

fi has aequationes

$$i=0 \quad I=0$$

$$i=1 \quad B=bb-ab-aa=0$$

$$i=2 \quad D=b^4-2ab^3-3aabb^2+a^3b+a^4=0$$

$$i=3 \quad F=b^6-3ab^5-6aabb^4+4a^3b^3+5a^4bb^2-a^5b-a^6=0$$

$$i=4 \quad H=b^8-4ab^7-10aabb^6+10a^3b^5+15a^4bb^4-6a^5b^3-7a^6b^2+a^7b+a^8=0$$

vnde concludimus, in genere fore:

$$0=b^{2i}-iab^{2i-1}-\frac{i(i-1)}{1 \cdot 2}a^2b^{2i-2}+\frac{i(i-1)}{1 \cdot 2 \cdot 3}a^3b^{2i-3}+\frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3 \cdot 4}a^4b^{2i-4}-\frac{i(i-1)(i-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^5b^{2i-5}-\frac{i(i-1)(i-4)(i-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}a^6b^{2i-6} \text{ etc.}$$

Pro altero casu, quo  $n=2i+1$ , habebimus:

fi has aequationes

$$i=0 \quad A=b-a=0$$

$$i=1 \quad C=b^3-2abb-aab+a^3=0$$

$$i=2 \quad E=b^5-3ab^4-3aabb^3+4a^3b^2+a^4b-a^5=0$$

$$i=3 \quad G=b^7-4ab^6-6aabb^5+10a^3b^4+5a^4b^3-6a^5b^2-a^6b+a^7=0$$

vnde concludimus in genere fore:

$$0=b^{2i+1}-(i+1)ab^{2i}-\frac{i(i+1)}{1 \cdot 2}a^2b^{2i-1}+\frac{i(i+1)(i+2)}{1 \cdot 2 \cdot 3}a^3b^{2i-2}-\frac{(i-1)i(i+1)(i+2)}{1 \cdot 2 \cdot 3 \cdot 4}a^4b^{2i-3}-\frac{(i-1)(i+1)(i+2)(i+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^5b^{2i-4}+\frac{(i-2)(i-1)(i+1)(i+2)(i+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}a^6b^{2i-5} \text{ etc.}$$

quae forma, si ponamus  $n=2i-1$ , commodius ita exhibetur:

$$0=b^{2i-1}-iab^{2i-2}-\frac{i(i-1)}{1 \cdot 2}a^2b^{2i-3}+\frac{i(i-1)}{1 \cdot 2 \cdot 3}a^3b^{2i-4}+\frac{i(i-1)(i-2)}{1 \cdot 2 \cdot 3 \cdot 4}a^4b^{2i-5}-\frac{i(i-1)(i-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^5b^{2i-6}-\frac{i(i-1)(i-4)(i-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}a^6b^{2i-7} \text{ etc}$$

Coroll.

Coroll. 1.

34. Cum in formulis generalibus supra exhibitis poni debeat  $c=b$ , erit pro casu  $n=2i$  aequatio generalis:

$$\mathfrak{A}\left(\frac{2bb-aa+aa\sqrt{(aa-4bb)}}{2}\right)^i + \mathfrak{B}\left(\frac{2bb-aa-a\sqrt{(aa-4bb)}}{2}\right)^i = 0$$

existente  $\mathfrak{A} + \mathfrak{B} = 1$  et  $\mathfrak{A} - \mathfrak{B} = \frac{-a-2b}{\sqrt{(aa-4bb)}}$ , hincque aequatio nostra:

$$\left(\frac{a+2b}{\sqrt{(aa-4bb)}} - 1\right)\left(\frac{2bb-aa+aa\sqrt{(aa-4bb)}}{2}\right)^i = \left(\frac{-a-2b}{\sqrt{(aa-4bb)}} + 1\right)\left(\frac{2bb-aa-a\sqrt{(aa-4bb)}}{2}\right)^i$$

Coroll. 2.

35. Pro casu autem  $n=2i+1$ , ob  $c=b$ , ex §. 23. adipiscimur hanc aequationem:

$$\mathfrak{A}\left(\frac{2bb-aa+a\sqrt{(aa-4bb)}}{2}\right)^i + \mathfrak{B}\left(\frac{2bb-aa-a\sqrt{(aa-4bb)}}{2}\right)^i = 0$$

vbi est  $\mathfrak{A} + \mathfrak{B} = b-a$  et  $\mathfrak{A} - \mathfrak{B} = \frac{(b+a)(a-2b)}{\sqrt{(aa-4bb)}}$ ; ideoque

$$(b-a + \frac{(b+a)(a-2b)}{\sqrt{(aa-4bb)}})\left(\frac{2bb-aa+a\sqrt{(aa-4bb)}}{2}\right)^i =$$

$$\left(\frac{(b+a)(a-2b)}{\sqrt{(aa-4bb)}} - b+a\right)\left(\frac{2bb-aa-a\sqrt{(aa-4bb)}}{2}\right)^i$$

Coroll. 3.

36. Si pro  $i$  scribamus  $i-1$ , vt sit  $n=2i-1$ , haec oritur aequatio:

$$\left(\frac{a-2b}{\sqrt{(aa-4bb)}} + 1\right)\left(\frac{2bb-aa+a\sqrt{(aa-4bb)}}{2}\right)^i = \left(\frac{a-2b}{\sqrt{(aa-4bb)}} - 1\right)\left(\frac{2bb-aa-a\sqrt{(aa-4bb)}}{2}\right)^i$$

Tom. XI. Nou. Comm. N hinc



hinc autem prodeunt superiores aequationes per  $2b$  multiplicatae.

### Scholion.

37. Formae generales hic exhibitae a summa potestate ipsius  $b$  incipiunt; eadem vero etiam ita inuersae repraesentari possunt, vt a summa potestate ipsius  $a$  incipient. Ita  $x = \frac{1}{2} - \frac{1}{2}b$ ,  $y = \frac{1}{2} - \frac{1}{2}a$ , colligimus hanc aequationem:

$$\begin{aligned} 0 = & a^{2i} - (2i-1)a^{2i-2}b^2 + \frac{(2i-2)(2i-3)}{1 \cdot 2} a^{2i-4}b^4 \\ & - \frac{(2i-2)(2i-4)(2i-5)}{1 \cdot 2 \cdot 3} a^{2i-6}b^6 \text{ etc.} \\ & + a^{2i-1}b - (2i-2)a^{2i-3}b^3 + \frac{(2i-3)(2i-4)}{1 \cdot 2} a^{2i-5}b^5 \\ & - \frac{(2i-4)(2i-5)(2i-6)}{1 \cdot 2 \cdot 3} a^{2i-7}b^7 \text{ etc.} \end{aligned}$$

Pro casu autem posteriori, quo  $n = 2i + 1$ , istam:

$$\begin{aligned} 0 = & + a^{2i+1} - 2ia^{2i-1}b^2 + \frac{(2i-1)(2i-2)}{1 \cdot 2} a^{2i-3}b^4 - \frac{(2i-2)(2i-3)(2i-4)}{1 \cdot 2 \cdot 3} a^{2i-5}b^6 \\ & - a^{2i}b + (2i-1)a^{2i-2}b^3 - \frac{(2i-2)(2i-3)}{1 \cdot 2} a^{2i-4}b^5 + \frac{(2i-3)(2i-4)(2i-5)}{1 \cdot 2 \cdot 3} a^{2i-6}b^7 \text{ etc.} \end{aligned}$$

Verum notandum est, has expressiones tantum eo vsque continuari debere, quoad ad terminum euanescentem perueniatur, et sequentes, etiamsi non euanescant, tamen reici oportere, cui cautioni formae superiores non sunt obnoxiae, ex quo eae quoque ad casus, vbi  $i$  non est numerus integer, extendi possunt, vbi quidem aequatio serie infinita constabit.

Proble-

## Problema II.

38. Si in triangulo ifoscele ABC angulus Tab. I.  
verticalis B fit multiplus anguli ad basin A in ra- Fig. 7.  
tione  $n:1$ , vt fit  $B=nA$ , inuestigare relationem in-  
ter basin  $AC=b$  et latera  $BA=BC=a$ .

## Solutio.

Pofitis angulis ad basin  $A=C=\alpha$ , vt fit  
verticalis  $B=n\alpha$ , erit  $(n+2)\alpha=\pi$ , ideoque  $\alpha=\frac{\pi}{n+2}$   
et  $B=\frac{n\pi}{n+2}$ . In formulis ergo fupra inuentis poni  
debet  $c=a$ , ita vt iam fcala relationis fit  $bb, -aabb$ .  
Quare cafus iterum binos diftinguendo, prout  $n$  fue-  
rit numerus par vel impar, habebimus:

Pro cafu  $n=2i$ .

fi	has aequationes
$i=0$	$I=0$
$i=1$	$B=bb-2aa=0$
$i=2$	$D=b^4-3aabb=0$
$i=3$	$F=b^6-4a^2ab^4+2a^4bb=0$
$i=4$	$H=b^8-5a^2ab^6+5a^4b^4=0$
$i=5$	$K=b^{10}-6a^2ab^8+9a^4b^6-2a^6b^4=0$
$i=6$	$M=b^{12}-7a^2ab^{10}+14a^4b^8-7a^6b^6=0$
$i=7$	$O=b^{14}-8a^2ab^{12}+20a^4b^{10}-16a^6b^8+2a^8b^6=0$
$i=8$	$Q=b^{16}-9a^2ab^{14}+27a^4b^{12}-30a^6b^{10}+9a^8b^8=0$
	etc.

N 2

quae

quae ad has formas simpliciores reducuntur :

$$\begin{aligned}
 i=1 & \quad bb-2aa=0 \\
 i=2 & \quad bb-3aa=0 \\
 i=3 & \quad b^3-4aabb+2a^3=0 \\
 i=4 & \quad b^4-5aabb+5a^4=0 \\
 i=5 & \quad b^5-6aab^3+9a^3b^2-2a^5=0 \\
 i=6 & \quad b^6-7aab^4+14a^4b^2-7a^6=0 \\
 i=7 & \quad b^7-8aab^5+20a^4b^3-16a^5bb+2a^7=0 \\
 i=8 & \quad b^8-9aab^6+27a^4b^4-30a^5bb+9a^8=0
 \end{aligned}$$

etc.

Hic ergo iterum duos casus discerni convenit, prout numerus  $i$  fit par vel impar.

Si fit  $i=2\lambda-1$  et  $n=4\lambda-2$  erit aequatio:

$$\begin{aligned}
 0 = & b^{2\lambda} - 2\lambda aab^{2\lambda-2} + \frac{2\lambda(2\lambda-1)}{1 \cdot 2} a^2 b^{2\lambda-4} - \frac{2\lambda(2\lambda-4)(2\lambda-5)}{1 \cdot 2 \cdot 3} a^3 b^{2\lambda-6} \\
 & + \frac{2\lambda(2\lambda-5)(2\lambda-6)(2\lambda-7)}{1 \cdot 2 \cdot 3 \cdot 4} a^4 b^{2\lambda-8} - \text{etc.}
 \end{aligned}$$

et ordine inverso ita se habebit:

$$0 = a^{2\lambda} - \frac{\lambda\lambda}{1 \cdot 2} a^{2\lambda-2} b^2 + \frac{\lambda\lambda(\lambda-1)}{1 \cdot 2 \cdot 3 \cdot 4} a^{2\lambda-4} b^4 - \frac{\lambda\lambda(\lambda-1)(\lambda-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} a^{2\lambda-6} b^6 + \text{etc.}$$

Sin autem fit  $i=2\lambda$ , et  $n=4\lambda$ , erit aequatio:

$$\begin{aligned}
 0 = & b^{2\lambda} - (2\lambda+1)aab^{2\lambda-2} + \frac{(2\lambda+1)(2\lambda-3)}{1 \cdot 2} a^2 b^{2\lambda-4} - \frac{(2\lambda+1)(2\lambda-5)(2\lambda-7)}{1 \cdot 2 \cdot 3} a^3 b^{2\lambda-6} \\
 & + \frac{(2\lambda+1)(2\lambda-7)(2\lambda-9)(2\lambda-11)}{1 \cdot 2 \cdot 3 \cdot 4} a^4 b^{2\lambda-8} - \text{etc.}
 \end{aligned}$$

quae ordine inverso ita se habebit:

$$\begin{aligned}
 0 = & (2\lambda+1)a^{2\lambda} - \frac{(2\lambda+1)\lambda(\lambda+1)}{1 \cdot 2 \cdot 3} a^{2\lambda-2} b^2 + \frac{(2\lambda+1)\lambda(\lambda-1)(\lambda+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^{2\lambda-4} b^4 \\
 & - \frac{(2\lambda+1)\lambda(\lambda-1)(\lambda-4)(\lambda+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} a^{2\lambda-6} b^6 \text{ etc.}
 \end{aligned}$$

feu

feu per  $2\lambda + 1$  diuidendo hoc modo:

$$0 = a^{2\lambda} - \frac{\lambda(\lambda+1)}{2 \cdot 3} a^{2\lambda-2} b^2 + \frac{\lambda(\lambda-1)(\lambda+2)}{2 \cdot 3 \cdot 4 \cdot 5} a^{2\lambda-4} b^4 - \frac{\lambda(\lambda-1)(\lambda+4)(\lambda+3)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} a^{2\lambda-6} b^6 \\ + \frac{\lambda(\lambda-1)(\lambda-4)(\lambda-3)(\lambda+4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} a^{2\lambda-8} b^8 - \text{etc.}$$

Nunc igitur ad alterum casum progrediamur.

Pro casu  $n = 2i + 1$ .

fi	erit aequatio
$i = 0$	$b - a = 0$
$i = 1$	$b^2 - abb - aab = 0$
$i = 2$	$b^3 - ab^2 - 2aab^2 + a^2bb = 0$
$i = 3$	$b^4 - ab^3 - 3aab^3 + 2a^2b^2 + a^3b = 0$
$i = 4$	$b^5 - ab^4 - 4aab^4 + 3a^2b^3 + 3a^3b^2 - a^4b = 0$
$i = 5$	$b^6 - ab^5 - 5aab^5 + 4a^2b^4 + 6a^3b^3 - 3a^4b^2 - a^5b = 0$

quae reducuntur ad has formas simpliciores:

$i = 0$	$b - a = 0$
$i = 1$	$bb - ab - aa = 0$
$i = 2$	$b^3 - abb - 2aab + a^2 = 0$
$i = 3$	$b^4 - ab^3 - 3aab^2 + 2a^2b + a^3 = 0$
$i = 4$	$b^5 - ab^4 - 4aab^3 + 3a^2bb + 3a^3b - a^4 = 0$
$i = 5$	$b^6 - ab^5 - 5aab^4 + 4a^2b^3 + 6a^3b^2 - 3a^4b - a^5 = 0$

vnde in genere concluditur fore:

$$0 = -ab^i + (i-1)a^2b^{i-2} - \frac{(i-1)(i-2)}{1 \cdot 2} a^3b^{i-4} + \frac{(i-2)(i-3)(i-4)}{1 \cdot 2 \cdot 3} a^4b^{i-6} \\ - \frac{(i-3)(i-4)(i-5)}{1 \cdot 2 \cdot 3} a^5b^{i-8} + \frac{(i-4)(i-5)(i-6)(i-7)}{1 \cdot 2 \cdot 3 \cdot 4} a^6b^{i-10} - \text{etc.}$$

# 102 PROPRIETATES TRIANGVLORVM.

Inuerſe autem duos caſus contemplari conuenit:

I. Si  $i = 2(\lambda - 1)$ , et  $n = 4\lambda - 3$ , erit

$$0 = a^{2\lambda-1} - \frac{\lambda}{1} a^{2\lambda-2} b^2 + \frac{\lambda(\lambda-1)}{1 \cdot 2} a^{2\lambda-3} b^4 - \frac{\lambda(\lambda-1)}{1 \cdot 2 \cdot 3} a^{2\lambda-4} b^6 + \frac{\lambda(\lambda-1)(\lambda-2)}{1 \cdot 2 \cdot 3 \cdot 4} a^{2\lambda-5} b^8 \text{ etc.}$$

II. Si  $i = 2\lambda - 1$ , et  $n = 4i - 1$ , erit

$$0 = a^{2\lambda} + \frac{\lambda}{1} a^{2\lambda-1} b - \frac{\lambda(\lambda+1)}{1 \cdot 2} a^{2\lambda-2} b^2 + \frac{\lambda(\lambda+1)}{1 \cdot 2 \cdot 3} a^{2\lambda-3} b^4 - \frac{\lambda(\lambda+1)(\lambda+2)}{1 \cdot 2 \cdot 3 \cdot 4} a^{2\lambda-4} b^6 \text{ etc.}$$

ſicque pro omnibus caſibus, quibus  $n$  eſt numerus integer, aequationes inter latera  $a$  et  $b$  eruimus.

SOLV-